

# NON-GAUSSIAN DISTRIBUTION OF GALAXIES GRAVITATIONAL FIELDS

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## RESUMEN

## ABSTRACT

We perform a theoretical analysis of the observational relation between angular momentum and mass (richness) of the galaxy clusters. For that we calculate the distribution function of astronomical objects (like galaxies and/or smooth halos of different kinds) gravitational fields due to their tidal interaction. Within the statistical method of Chandrasekhar we are able to show that the distribution function is determined by the form of interaction between objects and for multipole (tidal) interaction it is never Gaussian. Our calculation permits to demonstrate that alignment of galaxies angular momenta increases with the cluster richness. The specific form of the corresponding dependence is due to assumptions made about cluster morphology.

*Key Words:* Galaxies: Clusters — Galaxies: General

## 1. INTRODUCTION

In the models of galaxies and their structures formation the distribution of gravitational fields of their constituents play the decisive role. Many scenarios of such formation have been around for some time (Peebles 1969; Sunyaew & Zeldovich 1972; Zeldovich 1970; Doroshkevich 1973; Shandarin 1974; Dekel 1985; Efstathiou & Silk 1983). Under the influence of new observational data, these scenarios are constantly being revised and improved, see (Shandarin et al. 2012; Giahi-Saravani & Schäfer 2014) and references therein for latest discussion. The main controversy here is how galaxies acquire their angular moments, which render subsequently to those of galaxy clusters and larger structures. On the other hand, this angular moment acquisition is intimately related to the above gravitational fields distribution.

The commonly accepted model of the Universe is spatially flat homogeneous and isotropic  $\Lambda$ CDM model. The galaxy clusters in this model are formed as a result of adiabatic and almost scale invariant Gaussian fluctuations (Silk 1968; Peebles & Yu 1970; Sunyaew & Zeldovich 1970). This assumption is the base of the so-called hierarchical clustering model (Doroshkevich 1970; Dekel 1985; Peebles 1969), the most popular scenario of galaxies formation. Note, however, the presence of the models with non-Gaussian initial fluctuations, see (Bartolo et al 2004) and references therein. This non-Gaussianity, however, has been postulated in certain form rather than calculated. On the other hand, the non-Gaussian distributions can be obtained

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from initial Gaussian ones as a result of time evolution in the generalized stochastic models, where probability distribution functions (pdf's) are obtained from the solutions of the differential equations of Fokker-Planck type with so-called fractional derivatives (Garbaczewski & Stephanovich 2009; Garbaczewski et al 2011). In other words, the initial Gaussian fluctuations (if any) may become non-Gaussian as a result of primordial, fast time evolution. After it, the slower evolution, dictated by the  $\Lambda$ CDM scenario, occurs. Although here we do not present the details of this primordial time evolution, one of the aims of the present paper is to draw attention to the method, which permits to calculate the non-Gaussian distribution function, based solely on the form of interaction between astronomical objects. This distribution function is a terminal function for above initial fast time evolution process.

In hierarchical clustering type of scenarios, the large scale structure forms from bottom to top as a consequence of gravitational interactions between the constituents. This means that the smaller structures like galaxies are formed first with their subsequent merger into larger clusters. In this case, the galaxies spin angular momenta arise as a result of tidal interaction with their neighbours (Schäfer 2009). While in the originally hierarchical clustering scenarios, the completely random distribution of galaxies angular momenta has been obtained, it has been shown later that the local tidal shear tensor can cause a local alignment of their rotational axes (Catelan & Theuns 1996a,b; Lee & Pen 2002; Navarro et al 2004).

The above tidal torque mechanism has an opposite idea, constructed on the base of Zeldovich pancake model (Sunyaew & Zeldovich 1972; Doroshkevich 1973; Shandarin 1974). In this model, the structures in the Universe form from top to bottom. The crucial role here plays a magneto-hydrodynamic shock wave which makes the large structure to fragment. This shock wave arise as the result of asymmetrical collapse of a large structure and also imparts galaxies with spin angular momentum. The model predicts a coherent, non-random spatial orientation of galaxy planes with the galaxies rotational axes to be parallel to the main plane of a large structure.

In the model of primordial turbulences, (von Weizsäcker 1951; Gamow 1952; Ozernoy 1978; Efstathiou & Silk 1983) the spin angular momentum is a remnant of the primordial whirl so that the rotational axes of galaxies would be perpendicular to the main plane of the structures.

It had been pointed out in Refs. (Gamow 1946; Goedel 1949) and later in Ref. (Collins & Hawking 1973), that if the Universe is rotating, then the galaxies angular momentum is a consequence of its conservation in a rotating Universe. At that time, the argument against was that this model predicts the galaxies rotational axes alignment, which had not been confirmed observationally (see Ref. (Godłowski 2011) for details). Based on this idea, authors (Li 1998) proposed a model involving galaxy formation in a rotating Universe.

We emphasize, that simple picture, where each of the above approaches (primordial turbulences, hierarchical clustering and Zeldovich pancakes) pre-

dict different ways of galaxies rotational axes ordering is not completely true. The point is that in each of the above models including hierarchical clustering, the phase with shock wave can appear. Latter phase is usually accompanied by the collapse of structures or substructures (Melott & Shandarin 1989; Sahni et al 1994; Paulus & Melott 1995; Mo et al 2005; Shandarin et al. 2012), which may generate the rotational axes ordering. Apparently, the scale of such orientation is different in different models. For instance, in the Bower's scenario (Bower 2005), we do not have hierarchical clustering for all scales of masses. Instead, we have anti - hierarchical clustering in the small scales as tidal interaction effects yield Zeldovich pancake - like objects emergence (Zeldovich 1970) rather than spherically collapsing haloes. There is, however, a fundamental difference with above classical pancake scenario. Namely, the anti-hierarchical clustering is local as it occurs in small scale.

The model of hierarchical clustering is the only model explicitly taking into account the dark matter existence. The Li model has been originally considered as dust fluid, however, nothing prevents to introduce the dark matter as a background. As a result, in this model, the dark matter is not interacting with observable matter in any other way than gravitational forces. In the remaining models, namely primordial turbulences and Zeldovich pancake model, the only dust component has been considered so that there are no clear and successful attempts to introduce the dark matter there. Therefore, we exclude both models from the present investigation.

Theoretical models of galaxy formation have problems with explaining the observational dependence between structure angular momentum and its mass. The dependence can be seen only in the tidal torque scenario (Heavens & Peacock 1988; Hwang & Lee 2007; Noh & Lee 2006a,b) and in Li model (Li 1998; Godlowski et al 2005). The remaining models do not anticipate such dependence.

Comparing the two models, we should note that Li model needs a global or at least large scale rotation of the Universe. Li studied the correlation between the angular momentum and the mass of spiral galaxies and he estimated the rotation of the Universe to be close to the value obtained by Birch (Birch 1982). However, the obtained value is too large compared to observed anisotropy in CMBR. Therefore, in the present paper, we consider the Tidal Torque scenario only.

In the present work we perform the comprehensive theoretical analysis of the influence of tidal interaction between astronomical objects on the larger (then initial constituents) structures formation. For that we use the statistical method of Chandrasekhar (Chandrasekhar 1943), where we account also for dark matter haloes. The statistical method (Chandrasekhar 1943) permits to calculate the distribution functions of gravitational fields and angular momenta of stellar components. Our main result is that in the stellar systems with multipole (tidal) gravitational interaction, the derived distribution function cannot be Gaussian but rather belongs to the family of so-called "heavy-tailed distributions", see, e.g., Refs. (Garbaczewski & Stephanovich

2009; Garbaczewski et al 2011; van Kampen 1981) and references therein for details. As we have mentioned above, the obtained non-Gaussian pdf is a result of fractional time evolution for initial Gaussian fluctuations. This pdf, in turn, can be used to calculate the distribution of virtually any quantity (like angular momentum) of the astronomical objects (not only galaxy clusters but smooth component like haloes, which mass dominate the total mass of the cluster (Kravtsov & Borgani 2012)) in any (linear or nonlinear) Eulerian approach. Moreover, choosing the cosmology on the base of corresponding Friedmann equation, our result permits to obtain the time evolution of the distribution function of angular momenta and its mean value  $L$ . In our approach, we can also derive the well-known empirical relation between mean galaxy cluster moment  $L$  and its mass  $M$ ,  $L \sim M^{5/3}$ .

## 2. DISTRIBUTION FUNCTION OF GRAVITATIONAL FIELDS

To consider the tidal interaction in the ensemble of galaxies and their clusters, we regard both luminous and dark matter as the Newtonian self-gravitating dust fluid ( $p = 0$ ), embedded in the Friedmann - Lemaître - Robertson - Walker Universe. The tidal (shape distorting) interaction between the astronomical objects can be derived by the multipole expansion of the Newtonian interaction potential between fluid elements (Poisson 1998). Limiting ourselves to quadrupolar term, we write the Hamiltonian (total classical energy) of interaction between above elements (which equally well can belong to the luminous or dark matter) in the form

$$\mathcal{H} = -G \sum_{ij} Q_i m_j V(\mathbf{r}_{ij}), \quad V(\mathbf{r}) = \frac{1}{2} \frac{3 \cos^2 \theta - 1}{r^3}, \quad (1)$$

where  $G$  is the gravitational constant,  $Q_i$  and  $m_i$  are, respectively, the quadrupole moment and mass of  $i$ -th object,  $r_{ij} \equiv |\mathbf{r}_{ij}|$ ,  $\mathbf{r}_{ij} = \mathbf{r}_j - \mathbf{r}_i$  is a relative separation between objects and  $\theta$  is the apex angle. Note that expression (1) generalizes the two-particle result of Ref. (Poisson 1998) on the ensemble of  $N$  objects, splitting the interactions between them to that in pairs, see Appendix A for details. Such splitting is usual, for instance in the theory of magnetism, where the interacting spins ensemble is represented by the sum of all possible pair interactions between particles  $i$  and  $j$  like  $123 = 12 + 13 + 23$ , see, e.g. Ref. (Mattis 2007).

The Hamiltonian function (1) describes the pairwise, shape-distorting interaction between the astronomical objects. Namely, this interaction distorts the shape of a given  $i$ -th object, which alters its density field  $\rho_i(\mathbf{r})$ . As the objects have random shapes, their masses  $m_i$  and quadrupole moments  $Q_i$  vary randomly likewise the gravity field  $\mathbf{E}_{quad}$  from these quadrupoles. Latter field is a gradient of the potential (1) (divided by mass  $m$ ) and has the form

$$\mathbf{E}_{quad}(\mathbf{r}) = \mathbf{i}_r E_0 \frac{3 \cos^2 \theta - 1}{r^4}, \quad (2)$$

where  $E_0 = GQ/2$  and  $\mathbf{i}_r$  is the unit vector in radial direction.

According to statistical method of Chandrasekhar, the distribution function of random quadrupolar fields reads (Chandrasekhar 1943)

$$f(\mathbf{E}) = \overline{\delta(\mathbf{E} - \mathbf{E}_i)}, \quad (3)$$

where  $\delta(x)$  is Dirac  $\delta$  - function,  $\mathbf{E}_i \equiv \mathbf{E}_{quad}(\mathbf{r}_i)$  is given by the expression (2) and bar means the averaging over spatial (and any other possible) disorder. Note that if all objects in the ensemble are similar (no randomness), the distribution function is just delta-peak, centered at the field  $\mathbf{E}_i$ . The disorder broadens this delta-peak, giving rise to "bell-shaped" continuous probability distribution, see Refs. (Stephanovich 1997; Semenov & Stephanovich 2002, 2003) and references therein, where the statistical method had been applied to disordered solids.

The explicit averaging in (3) is performed with the help of the integral representation of Dirac  $\delta$  - function, see Ref. (Stephanovich & Godłowski 2015) for details. The idea is that the mass and quadrupole moment of the object in the volume  $V$  obey the uniform distribution with probability density equal to  $1/V$ . In this case we introduce the number of objects  $N$  so that in the limit  $N \rightarrow \infty$  and  $V \rightarrow \infty$ , their density  $n = N/V$  remains constant. Final expression for the distribution function (3) reads (Stephanovich & Godłowski 2015)

$$f(\mathbf{E}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} e^{i\mathbf{E}\boldsymbol{\rho} - F(\rho)} d^3\rho, \quad (4)$$

$$F(\rho) = n \int_V \left[ 1 - \frac{\sin \rho E(\mathbf{r})}{\rho E(\mathbf{r})} \right] d^3r. \quad (5)$$

We see that  $F(\rho)$  is indeed the characteristic function for random gravitational fields distribution. Note that characteristic function  $F(\rho)$  depends only on modulus  $\rho$  and not its angles. This will result (see Eq. (8) below) in the only field modulus dependence of pdf of random gravitational fields. The reason is that we take only  $zz$  component of quadrupolar field in Eq.(2). If we need the complete (i.e. including its possible angular dependence) distribution function of vector  $\mathbf{E}$ , we should account for complete tensor structure of Hamiltonian (1)  $\mathcal{H} = -G \sum_{ij\alpha\beta} Q_{i\alpha\beta} m_j V_{\alpha\beta}(\mathbf{r}_{ij})$ ,  $\alpha, \beta = x, y, z$ . Such account, while not changing our conclusions qualitatively (Stephanovich 1997; Semenov & Stephanovich 2002, 2003) (and in many cases quantitatively, see below), will make problem to be tractable only numerically. At the same time our present approach permits to have analytical insights into the problem (for example investigate the implication of non-Gaussian character of distribution function of gravitational fields), which is good starting point for future numerical simulations. One more justification of the radial distribution is the results of numerical simulations in halo model (Schneider & Bridle 2010), where the axes of galaxies embedded in dark matter halo, were preferentially radially oriented.

Moreover, the spin angular momentum is known only for very few galaxies and other structures. Therefore, instead of the angular momentum by itself, the orientation of galaxies in each cluster is usually studied. This is also the reason that here we are interested primarily in the radial distribution of galaxies' characteristics.

In more realistic models of galaxy clustering we can assume that the objects (e.g. galaxies) density is not a constant but rather depends on their separation  $n = n(\mathbf{r})$ . The other factor, which may improve the coincidence with observational results is to consider the galaxy clustering within a model of inhomogeneous distribution of masses (and/or quadrupolar moments) in the ensemble. This can be done along the lines of Ref. (Chandrasekhar 1943), where the distribution function of masses  $\tau(m)$  had been introduced.

We note here that distribution function  $f(\mathbf{E})$  (4) in general case is much more complicated than Gaussian. We had shown in Ref. (Stephanovich & Godłowski 2015) that for multipole interaction between astronomical objects, the function (4) does not admit Gaussian limit. To demonstrate that, we note that Gaussian limit corresponds to large density  $n \rightarrow \infty$ , giving in turn  $\rho \rightarrow 0$  in (5) (Stephanovich 1997; Semenov & Stephanovich 2003; Stephanovich & Godłowski 2015). In first nonvanishing (Gaussian) approximation in small  $\rho$  this procedure yields

$$F_G(\rho) = \frac{n\rho^2}{6} \int_V E^2(\mathbf{r}) d^3r, \quad (6)$$

generating after substitution of (2)

$$F_G(\rho) = \frac{1}{3} \pi n \rho^2 \int_0^\pi (3 \cos^2 \theta - 1)^2 \sin \theta d\theta \int_0^\infty \frac{dr}{r^6}. \quad (7)$$

It is seen that the integral is divergent at small  $r$ , which shows that the distribution function cannot be Gaussian. Note that in disordered solids the Gaussian limit exists (i.e. the integral (7) converges) due to presence of short range terms  $\sim \exp(-r/r_c)$  ( $r_c$  defines the range of interaction) in the interaction potential between dipoles or spins.

The calculation of  $F(\rho)$  (5) generates following explicit form of  $f(E)$  (Stephanovich & Godłowski 2015)

$$f(E) = \frac{1}{2\pi^2 E} \int_0^\infty \rho e^{-\alpha \rho^{3/4}} \sin \rho E d\rho, \quad (8)$$

$$\alpha = 2\pi n \cdot 0.41807255 \cdot E_0^{3/4}.$$

The expression (8) is the main theoretical result of the present paper. The distribution function (8) depends parametrically on the objects (i.e. both luminous and dark matter) density  $n$ , and on average quadrupole moment  $Q$ .

The normalization condition for distribution function (8) looks like

$$4\pi \int_0^\infty E^2 f(E) dE = 1. \quad (9)$$

As we have shown above in our approach, the distribution function of the gravitational fields cannot be Gaussian for multipole interaction potential between galaxies or any other astronomical objects including elements of dark matter halos. However, all previous theories postulated the distribution function in the Gaussian form rather than derived it. We mention here that non-Gaussian distribution have also been postulated rather than calculated (Bartolo et al 2004). In our opinion, non-Gaussian, heavy-tailed nature of the above distribution function captures the essential physics of the systems with long-range gravitational multipole interaction. Namely, the long-range interaction in such systems makes the objects (galaxies, their clusters and even the dark matter halos) to interact with each other also at very large separations. This, in turn, implies nonzero probabilities of such configurations, contrary to the case of Gaussian distribution, generated by short-range interactions. Below we will see the important implications of this fact.

To plot the function  $f(E)$ , we introduce the dimensionless variables  $\rho E = x$  and  $\beta = E/\alpha^{4/3}$ . In these variables the integral (8) renders to

$$f(\beta) = \frac{H(\beta)}{4\pi\beta^2\alpha^4}, \quad H(\beta) = \frac{2I(\beta)}{\pi\beta}, \quad (10)$$

$$I(\beta) = \int_0^\infty x \sin x \exp \left[ - \left( \frac{x}{\beta} \right)^{3/4} \right] dx. \quad (11)$$

The physical meaning of the function  $H(\beta)$  is that it gives the effective 1D distribution function of random gravitational fields. This is because the normalization condition for  $H(\beta)$  assumes one-dimensional form  $\int_0^\infty H(\beta) d\beta = 1$ , see (9). In this case, the average value of dimensionless random field  $\beta$  reads  $\bar{\beta} = \int_0^\infty \beta H(\beta) d\beta$ . The mean value  $\bar{\beta}$  exists if the integral  $H(\beta)$  is convergent. For that it is instructive to obtain the explicit expression for its asymptotics. At  $\beta \rightarrow 0$  we make substitution  $x/\beta = t$  to obtain from (11)  $H(\beta \rightarrow 0) \approx 48\beta^2/(3\pi)$ . At  $\beta \rightarrow \infty$  we expand Eq. (11) over small parameter  $1/\beta$ , which yields  $H(\beta \rightarrow \infty) = 2\Gamma(11/4) \cos \frac{\pi}{8} / (\pi\beta^{-7/4}) \approx 0.945972642\beta^{-7/4}$ , where  $\Gamma(x)$  is  $\Gamma$ -function (Abramowitz & Stegun 1964). Having asymptotics  $H(\beta)$ , we calculate those for  $f(\beta)$  with the help of relation (10):

$$f(\beta) = \begin{cases} \frac{4}{\pi^2\alpha^4}, & \beta \rightarrow 0 \\ \frac{0.945972642}{4\pi\alpha^4} \beta^{-15/4}, & \beta \rightarrow \infty. \end{cases} \quad (12)$$

The asymptotics (12) shows that  $f(\beta)$  does not depend on  $\beta$  at small  $\beta$  and decays at large  $\beta$ . The character of decay at large  $\beta$  shows that although normalization integral is convergent (the corresponding integrand decays at infinity as  $\beta^{-7/4}$ ), already first moment does not exist. This is a confirmation of the fact that function  $f(\beta)$  belongs to the class of heavy-tailed distributions.

### 3. DISTRIBUTION FUNCTION OF ANGULAR MOMENTA

To calculate the distribution function of angular momenta, we need a relation between the angular momentum  $\mathbf{L}$  of a galaxy and its gravitational

field  $\mathbf{E}_{quad}(\mathbf{r})$  (2). The expression for angular momentum components  $L_\alpha$  ( $\alpha = x, y, z$ ) has been derived perturbatively in small Lagrangian coordinate  $\mathbf{q}$ . The first order terms are defined by Eq. (11) of Ref. (Catelan & Theuns 1996a), while second order ones by Eq. (28) of the follow-up article (Catelan & Theuns 1996b). Both expressions have identical structure  $L_\alpha^{(i)} = f_i(t) \varepsilon_{\alpha\beta\gamma} E_{i\beta\sigma} I_{\sigma\gamma}$ ,  $\alpha, \beta, \gamma, \sigma = x, y, z$ , where index  $i = 1, 2$  denotes the order of perturbation theory,  $\varepsilon_{\alpha\beta\gamma}$  is Levi-Civita symbol,  $E_{\beta\sigma}$  are components of quadrupole (tidal) field (2) and  $I_{\sigma\gamma}$  are components of inertia tensor.

In order to calculate the distribution function of *modulus* of  $E$  (and subsequently  $L$ ), it is sufficient to take  $zz$  component in (2). If we need the complete distribution function of vector  $\mathbf{E}$ , we should account for complete tensor structure of Hamiltonian (1)  $\mathcal{H} = -G \sum_{ij\alpha\beta} Q_{i\alpha\beta} m_j V_{\alpha\beta}(\mathbf{r}_{ij})$ ,  $\alpha, \beta = x, y, z$ . Also, as  $\mathbf{L}$  depends on  $t$  by means of above functions  $f_i(t)$ , the distribution function will be time dependent.

The components of the gravitational tidal (shear) field  $\mathbf{E}$  are different in the first and second orders of perturbation theory, but they have the same structure (the expression above) where the order is governed by the index  $i = 1, 2$ . Taking into account the symmetry relations  $I_{ab} = I_{ba}$  and  $E_{ab} = E_{ba}$  and leaving only  $E_{zz}$ , we obtain  $L_x = -b(t)E_{zz}I_{yz}$ ,  $L_y = b(t)E_{zz}I_{xz}$ ,  $L_z = 0$ ,  $L = \sqrt{L_x^2 + L_y^2 + L_z^2} = L_0 E$ ,  $L_0 = L_0(t) = f_i(t) \sqrt{I_{xz}^2 + I_{yz}^2}$ . This expression constitutes linear relation between angular momentum and tidal field moduli both in linear ( $i=1$ ) and nonlinear ( $i=2$ ) regimes. As the above relation between gravitational field modulus and angular momentum is linear, the shape of distribution function of angular moments  $f(L)$  repeats that of gravitational fields. The explicit expression for  $f(L)$  can be derived using known relation from the theory of probability  $f(L) = f[E(L)] \left| \frac{dE}{dL} \right|$ , which yields

$$f(L) = \frac{1}{2\pi^2 L} \int_0^\infty \rho e^{-\alpha \rho^{3/4}} \sin\left(\rho \frac{L}{L_0(t)}\right) d\rho, \quad (13)$$

where  $L_0(t)$  is defined above. Dimensionless variables  $\rho(L/L_0) = x$ ,  $\lambda = L/(L_0 \alpha^{4/3})$  generate the following pair of functions similar to the case of gravitational fields distribution

$$f(\lambda) = \frac{H(\lambda)}{4\pi \lambda^2 \alpha^4 L_0}, \quad H(\lambda) = \frac{2I(\lambda)}{\pi \lambda}, \quad (14)$$

where  $I(\lambda)$  is defined by the expression (11).

The effective 1D distribution of fields or momenta is reported in left panel of Fig. 1. It is seen that while initial 3D function  $f(\lambda)$  decays monotonically (right panel), this function has characteristic bell shape and is asymmetric. Note, that as the initial equation (1) allows for the interaction everyone with everyone astronomical objects in an ensemble, it considers naturally the interaction with surrounding structures and dark matter haloes also. This fact renders the distribution functions of gravitational fields (10) and angular momenta (14) to account not only for the isolated cluster regions, but long-range interactions with surrounding structures as well. To be specific, the narrow



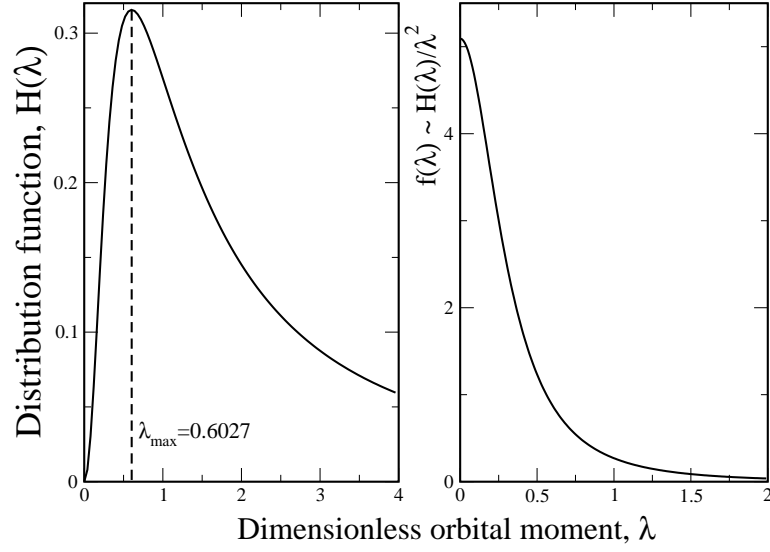


Fig. 1. Left panel. The effective 1D distribution function  $H(\lambda)$  (14). The distribution function (11) has the same shape. Dashed line shows the argument  $\lambda_{max}$ , corresponding to maximum of  $H(\lambda)$ . Right panel shows 3D distribution function  $4\pi\alpha^4 L_0 f(\lambda) = H(\lambda)/\lambda^2$ .

peak of distribution function in left panel of Fig.1 stems from the closely situated cluster region, while its long tails stem from the long-range (quadrupole) interaction with surrounding structures. In other words, the interaction with surrounding structures is essential (and our distribution functions take this fact into account) as the interaction between objects in stellar ensembles have long-range multipole character.

Asymptotics (12) shows that the integral for its mean value diverges. It is well - known (see, e.g., (van Kampen 1981)) that for the distribution functions, which decay slowly at infinities, the corresponding mean value can be approximately estimated as the maximum of such function. In this spirit we calculate  $\lambda_{max}$ , corresponding to the maximum of distribution function  $H(\lambda)$ , see Fig. 1 for details. The analysis of  $\lambda_{max}$  in dimensional units makes possible to derive some useful relations, which earlier had been taken as empirical ones. To consider the characteristics of galaxies, i.e. luminous matter, here we use the ideas of halo model (Schneider & Bridle 2010), which states that galaxies (i.e. "pieces" of luminous matter) are embedded in the dark matter haloes so that observable characteristics of galaxies like their angular momentum emerge from the mass and hence gravitational field of dark matter. Also, as the galaxies and their clusters reside in the larger structures like voids and filaments, the gravitational field of latter large objects also influence galaxies, see, e.g. Ref. (Joachimi et al 2015). As our distribution function (14) takes these effects into account by virtue of model (1), our subsequent calculations of mean angular momentum of the galaxies take above effects into account.

Let us first consider the simplest possible CDM cosmology in the first order of perturbation theory, where  $a(t) = D(t) = (t/t_0)^{2/3}$  (Doroshkevich 1970) so that  $L_0 = \frac{2I}{3t_0}\tau$ ,  $\tau = t/t_0$ ,  $I = \sqrt{I_{xz}^2 + I_{yz}^2}$ . The solution of the equation  $dH/d\lambda = 0$  reads  $\lambda_{max} = 0.602730263$ , giving in dimensional units

$$\begin{aligned} L_{max} &= 0.7281884n^{4/3}\frac{t}{t_0^2}GIQ \approx \\ &\approx \kappa n^{4/3}\frac{t}{t_0^2}GR^4m^2, \end{aligned} \quad (15)$$

where  $n = N/V$ ,  $\kappa$  is a constant of order unity. To derive the equation (15), we estimate galaxy quadrupole moment  $Q$  and its mean inertia moment  $I$  as being proportional to  $mR^2$ , where  $m$  is galaxy mass and  $R$  is its mean radius. If we represent volume  $V$  as  $V = R^3$ , then  $R$  cancels down in Eq. (15) so that  $L_{max} \sim (t/t_0^2)m^2N^{4/3}$ . Then, we introduce the mass of a galaxy cluster  $M = mN$  and obtain

$$L_{max} \sim \frac{t}{t_0^2}M^{5/3}\left(\frac{m}{N}\right)^{1/3} \equiv \frac{t}{t_0^2}M^{5/3}\frac{\rho^{1/3}}{n^{1/3}}, \quad (16)$$

where  $\rho = m/V$  is a mass density and  $n = N/V$  is galaxies density. Following (Catelan & Theuns 1996a), we assume that mass density  $\rho$  is a function of time, defined by Friedmann equation in CDM model  $\dot{a}/a = H_0 = \sqrt{8\pi G\rho/3}$ , where  $H_0$  is the Hubble constant. This generates the dependence  $\rho \propto t^{-2}$ ,

which, being substituted to (16), yields

$$L_{max} \sim \frac{t^{1/3}}{t_0^2} \frac{M^{5/3}}{n^{1/3}} \sim t^{1/3} M^{5/3}. \quad (17)$$

To derive Eq. (17), we assume that  $n = \text{const.}$  We see that equations (16) and (17) recover the expression (27) of (Catelan & Theuns 1996a), giving the theoretical derivation of well-known (see, e.g., (Catelan & Theuns 1996a) and references therein) empirical relation between mean angular momentum of galaxies ensemble (galaxy clusters) and their mass  $L_{max} \sim M^{5/3}$ .

There is also other approach to interpretation of the dependence of  $L_{max}$  on stellar parameters. Namely, suppose that volume  $V = R_A^3$ , where  $R_A$  is a mean cluster radius, proportional to the autocorrelation radius (see (Longair 2008) and references therein). Although  $n$  is still a constant for any particular cluster, it varies from cluster to cluster with increasing richness  $N$ . In this case we may rewrite  $N = M/m$  to obtain the alternative (to Eq. (17)) form of expression for  $L_{max}$

$$L_{max} \sim \frac{t}{t_0^2} \left( \frac{R}{R_A} \right)^4 m^{2/3} M^{4/3}, \quad (18)$$

which does not contain  $\rho$ .

It is instructive to comment on time dependence  $L_{max}(t)$  in Eq. (18). On the first sight, it follows from (18) that  $L_{max} \sim t$ , but the problem complicates a lot by the intricate time dependence of the quantities  $R$  and  $R_A$  (Longair 2008). We plan to study this question in subsequent publications.

It is clear from the equation (15) that mean orbital moment of a galaxy increases with the number of galaxies  $N$  and it is proportional to  $N^{4/3}$ . Moreover, even in the model with constant galaxies density  $n$ , number (richness)  $N$  varies from cluster to cluster so that the dependence  $L_{max}(N) = \kappa_2 N^{4/3}$  holds and shows that the systems with larger number of galaxies  $N$  have larger angular momenta.

#### 4. TIME DEPENDENCE OF DISTRIBUTION FUNCTION IN $\Lambda$ CDM MODEL

Time evolution of distribution function (14) relies on explicit dependences  $f_1(t)$  and  $f_2(t)$ . The functions  $f_1(t) = a^2(t)\dot{D}(t)$  and  $f_2(t) = \dot{E}(t)$  (dot means time derivative) are calculated from the differential equations, derived in  $i$ -th order of perturbation theory (Bouchet et al 1992):

$$t_0^2 \ddot{D}(t) + a(t)D(t) = 0, \quad (19)$$

$$t_0^2 \ddot{E}(t) + a(t)E(t) = -a(t)D(t)^2, \quad (20)$$

where  $0 \leq t < \infty$  is dimensional physical time and  $t_0$  is some characteristic time, depending of the cosmological model considered, see below. The dimensionless function (scale factor)  $a(t)$  is determined from the first Friedmann equation

$$\frac{H^2}{H_0^2} = \Omega_R a^{-4} + \Omega_M a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda. \quad (21)$$

Here  $H = \dot{a}/a$  is Hubble parameter ( $\dot{a} \equiv da/dt$ ),  $H_0$  is Hubble constant and  $\Omega_i$  ( $i = R, M, k, \Lambda$ ) are corresponding density parameters taken at present time, when  $a(t) = 1$ . Specifically,  $\Omega_R$  is radiation density,  $\Omega_M$  is matter (dark plus baryonic) density,  $\Omega_k$  is co-called spatial curvature density and  $\Omega_\Lambda$  is cosmological constant or vacuum density,  $\Omega_\Lambda = \Lambda/(3H_0^2)$ , where  $\Lambda$  is cosmological constant. For our calculations of distribution function of angular momenta, we will choose  $\Lambda$ CDM model of the Universe, keeping in the equation (21) only  $\Omega_M$  and  $\Omega_\Lambda$  terms,  $\Omega_M + \Omega_\Lambda = 1$ .

The further step is to determine the dependence  $L_0(t)$ . This generates substitution  $\lambda \rightarrow \lambda/f_i(\tau)$ , where  $\tau = t/t_0$  so that we have from (14)

$$H(\lambda, \tau) = \frac{2I(\lambda/f_i(\tau))}{\pi\lambda}, \quad i = 1, 2. \quad (22)$$

To obtain  $f_{1,2}(\tau)$  in  $\Lambda$ CDM model, we begin with the determination of  $a(t)$  from Friedmann equation (21), which reads in this case

$$\frac{da}{dt} = H_0 \sqrt{\Omega_\Lambda a^2 + \frac{1 - \Omega_\Lambda}{a}}. \quad (23)$$

The solution of the equation (23) has the form

$$a(t) = \alpha \sinh^{2/3}(t/t_0), \quad \alpha = \left( \frac{1 - \Omega_\Lambda}{\Omega_\Lambda} \right)^{1/3}, \quad (24)$$

$$t_0 = \frac{2}{3H_0\sqrt{\Omega_\Lambda}}.$$

Having the function  $a(t)$ , we can solve equation (19) numerically for  $D(\tau)$  and then determine the function  $f_1(\tau) = a^2(\tau)D'(\tau)$  ( $D' = dD/d\tau$ ). Accordingly, in the nonlinear regime, the function  $f_2(\tau) = E'(\tau)$  should be found numerically from the equation (20).

We note that functions  $f_2(\tau)$ , related to the second perturbative corrections, are negative. For instance, in Einstein - de Sitter model  $f_1(\tau) = (2/3)\tau$  and  $f_2(\tau) = (-4/7)\tau^{1/3} < 0$  (Doroshkevich 1970; Catelan & Theuns 1996b). The same result ( $f_2(\tau) < 0$ ) can be obtained numerically for  $\Lambda$ CDM model. Also, the maximum of the function (22) occurs at  $\lambda_{max}(\tau) \approx 0.602730263 f_i(\tau)$ , where  $\lambda_{max} \approx 0.602730263$  is a maximum of time - independent function  $H(\lambda)$  (14). Substitution of the negative function  $f_2(\tau) < 0$  into the integrand (22) generates the imaginary part, which does not change the form of  $H(\lambda, \tau)$  qualitatively. That is why everywhere we use the moduli of the functions  $f_2(\tau)$ .

The dependences  $H(\lambda, \tau)$  (22) for CDM (with above analytical expressions for  $f_i(\tau)$ ) and  $\Lambda$ CDM models are shown in the Fig. 2. It is seen that as time increases, the distribution function diminishes, while it grows to infinity at  $t \rightarrow 0$ . As time (figures near curves in Fig. 2) grows, the whole distribution function "blurs" as its maximum shifts towards large  $t$ . It is seen that "blurring" of distribution function at large times is much faster for  $\Lambda$ CDM model. Also, both in linear and nonlinear regime  $H(\lambda, \tau)$  grows with time, although

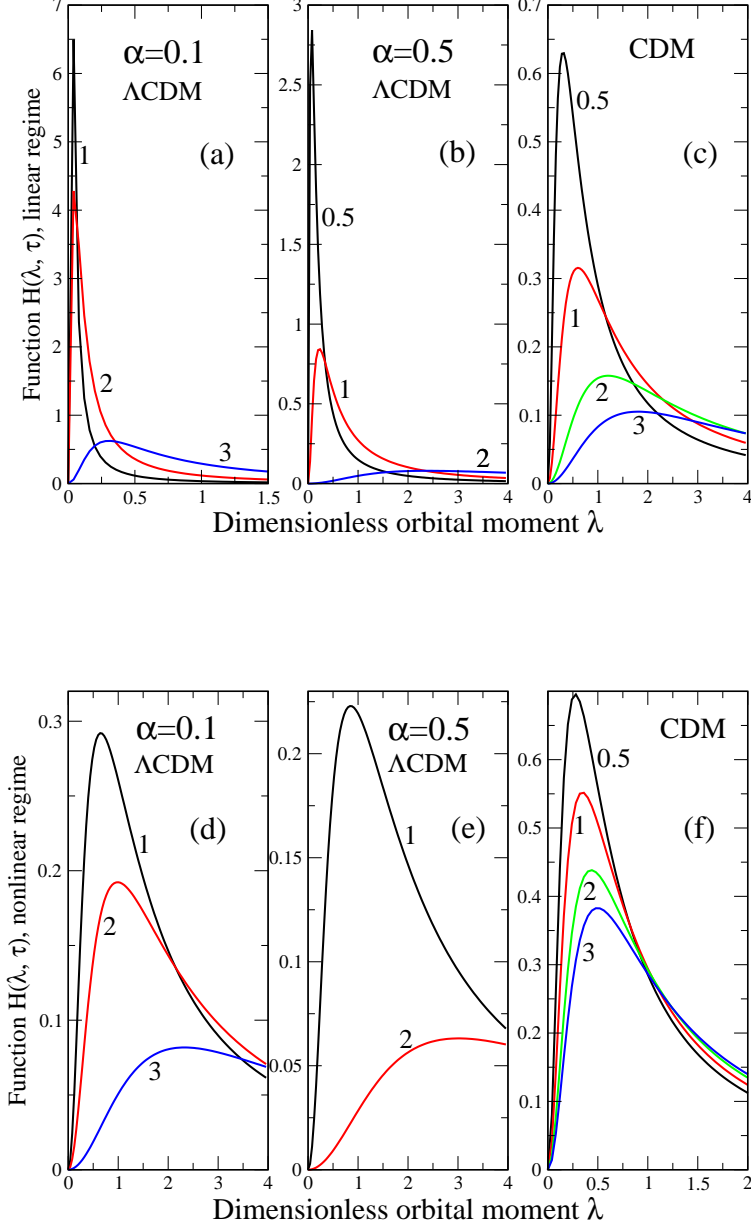


Fig. 2. Time evolution of effective 1D distribution function  $H(\lambda, \tau)$  in  $\Lambda$ CDM (with parameters  $\alpha = \left(\frac{1-\Omega_\Lambda}{\Omega_\Lambda}\right)^{1/3}$  shown in the panels) and CDM (panels (c) and (f)) models. Upper panels - linear regime, lower panels - nonlinear regime. Figures near curves correspond to dimensionless time  $\tau = t/t_0$ .

in  $\Lambda$ CDM model this growth is much faster. This is related to the fact that functions  $f_i(\tau)$  enter the exponent in the integrand (22). The comparison of upper and lower panels of Fig. 2 show that the behaviour of  $H(\lambda, \tau)$  is qualitatively similar in linear and nonlinear regimes of fluctuation growth. This means that for qualitative analysis we may safely use the linear regime. Additionally, as has been noticed in many references (see, e.g. (Catelan & Theuns 1996b) and references therein), that first order perturbation result (linear regime) corresponds to so-called Zeldovich approximation which is approximately valid also for nonlinear situation. This shows once more that for qualitative discussion of the time dependence of the distribution function  $H(\lambda, \tau)$  we can use the linear approximation with solution  $f_1(\tau)$ .

## 5. RELATION TO OBSERVATIONAL RESULTS. CONCLUSIONS

Our calculations demonstrate that although the gravitational interaction between stellar components (including dark matter halos) is of long-range multipole character, the observations (which we will discuss below) may evidence that there is additional short-range (like  $\sim \exp(-r/r_c)$  with range  $r_c$ ) interaction. This means that if the distance  $r$  between two objects (say galaxies) is smaller than  $r_c$ , they are correlated and have their orbital moments aligned. This is the case for the dense (rich) galaxy clusters, which, by this virtue, have high degree of orbital moments alignment. In the opposite situation of poor clusters, where the intergalaxy distance  $r > r_c$ , the long-range multipole interaction prevails so that there is no alignment of the orbital moments. The above statistical method accounts for this situation if we add the (empirical) short-range interaction term to the initial potential (2). In this case, the distribution function of random fields would depend on the average angular momentum  $L_{max} \equiv L_{av}$  (see (Stephanovich 1997; Semenov & Stephanovich 2003)) so that the self-consistent equation for  $L_{av}$

$$L_{av} = \int L(E) f(E, L_{av}) d^3E \quad (25)$$

can be derived. Here  $f(E, L_{av})$  is the distribution function, which substitutes the expression (8) in the case of inclusion of the possible short-range interaction term. In such case, for finite  $r_c$ , the distribution function decays at  $E \rightarrow \infty$  faster than (8) so that the integral (25) converges. As the total interaction potential includes both luminous and dark matter, the equation (25) permits to address the question about alignment of sub-dominant galaxies, when most of cluster angular momentum is in the smooth dark matter halo component. For instance, in the halo model (Schneider & Bridle 2010), when the luminous matter of galaxies is embedded in dark matter halo, this halo by virtue of its mass may mediate the intergalaxy interaction, adding possible short-range terms to it. The self-consistent equation (25) permits also to include the temperature into consideration (Semenov & Stephanovich 2002, 2003) and study the galaxies and their clusters (with respect to dark matter haloes) time evolution within  $\Lambda$ CDM model. Also, the combination

of stochastic models (Garbaczewski & Stephanovich 2009; Garbaczewski et al 2011) of primordial dynamics along with those of  $\Lambda$ CDM, most probably, would permit to answer (at least theoretically) the question if the galaxies are initially aligned in at the time of their formation, or such alignment is generated in some merger events, and how dark matter haloes influence (mediate) this alignment.

The examination of observational data regarding galaxies orientation in clusters have been carried out in Ref. (Godłowski et al 2010). The aim was to check if really the alignment of galaxies angular momenta increases with the cluster richness. For that, authors (Godłowski et al 2010) studied the spatial orientations of galaxies in the 247 optically selected rich (with at least 100 members in the considered area) Abell clusters taken from PF catalog (Panko 2006). The performed linear regression analysis makes possible to conclude that cluster angular momenta increase with their numerousness. It is clear at the same time, that relatively small statistical sample of 247 clusters, analyzed in Ref. (Godłowski et al 2010), does not permit to establish unambiguous correspondence of different dependences between angular momentum and richness of the structure. Namely, we were not able to discriminate definitely the linear dependence tested in Ref. (Godłowski et al 2010), the dependences  $L_{max} \sim M^{5/3}$  (equation (16) and Ref. (Catelan & Theuns 1996a)) and our result obtained in (18) that mean angular momentum is proportional to  $M^{4/3}$  (stemming from  $N^{4/3}$ ). However, such unambiguous discrimination would be possible if larger statistical sample of galaxy clusters is available. One should note that no evidence for rotation of galaxy clusters (see, e.g., (Regos & Geller 1989; Hwang & Lee 2007)), lead to the conclusion that their angular momentum is related primarily to the alignment of constituent galaxies spins. The above results, lack of alignment of the orientation of galaxies for group and poor galaxy clusters, and evidence for alignment for the rich clusters of galaxies (Ref. (Godłowski et al 2005), see also Ref. (Godłowski 2011) for later improved analysis) clearly shows that angular momentum of galaxy group and clusters increases with their richness. Observational data are in agreement with our theoretical results (mainly Eqs. (18) and (16) where we have shown that with reasonable assumption about cluster morphology the angular momentum of galaxy structures increase with their richness. The solution of equation (25) will permit to establish relation between the characteristics of possible short-range intergalaxy interaction and character of their spins alignment.

We finally note that the direct computer simulations of stellar ensembles are still quite computationally expensive to simulate realistic (i.e. sufficiently large) parts of the Universe. Hence it seems to be a good idea to put some effort into developing new theoretical models for galaxy alignments with respect to dark matter haloes and (possible) merger into larger structures like superclusters. Since galaxy morphology plays important role in this behavior, our approach, linking the galaxy shapes with their characteristics distribution (especially in view that it permits to calculate the non-Gaussian pdfs), will

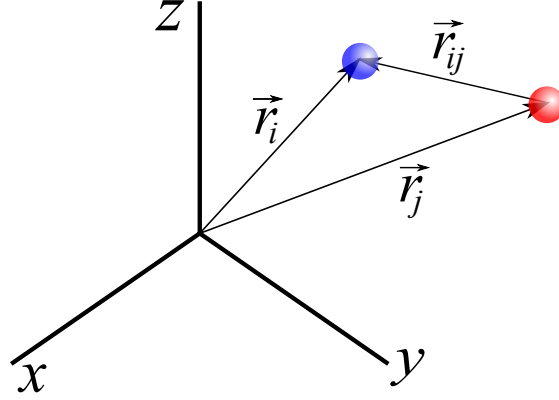


Fig. 3. The reference frame of the problem under consideration. Radius - vectors of galaxy (or dark matter halo element)  $i$  (blue ball) and  $j$  (red ball) ( $\mathbf{r}_i$  and  $\mathbf{r}_j$  respectively) as well as their difference  $\mathbf{r}_{ij}$  are shown.

improve the overall understanding, which can additionally be tested against observed galaxy shape distributions and alignments.

## APPENDICES

### A. SOME DETAILS OF OUR MODEL

Here we present some more details of our model, based on Hamiltonian (1). In this Hamiltonian, the explicit expression for  $i$ -th galaxy quadrupolar moment  $Q_i$  has the form (Poisson 1998)

$$Q_i = \int_{V_i} \rho_i(\mathbf{x}) |\mathbf{x}|^2 P_2(\mathbf{s} \cdot \mathbf{x}) d^3x, \quad (\text{A26})$$

where  $P_2(x) = (3x^2 - 1)/2$  is corresponding Legendre polynomial (Abramowitz & Stegun 1964),  $V_i$  is a volume of  $i$ -th galaxy,  $\rho_i(\mathbf{x})$  is a density of its mass.

The geometry of the problem under consideration is shown in Fig.3. It is seen first, that the origin is not related to any specific galaxy or other astronomical object. Rather, it is situated in the arbitrary point in the Universe. Although  $\mathbf{r}_{ij}$  is directed from one galaxy (in our case  $j$ ) towards another (in our case  $i$ ) it is by no means bounded to these galaxies. It simply means the difference in their radius - vectors, which connect the coordinates origin and position of each galaxy.

The Hamiltonian (1) can be identically rewritten through the interaction energy

$$\begin{aligned} \mathcal{H} &= -GM^2 \sum_i p_i m_i W_i, \\ W_i &= W(\mathbf{r}_i) = \sum_j m_j V(\mathbf{r}_{ij}) \equiv \\ &\quad \sum_j m_j V(\mathbf{r}_j - \mathbf{r}_i). \end{aligned} \quad (\text{A27})$$



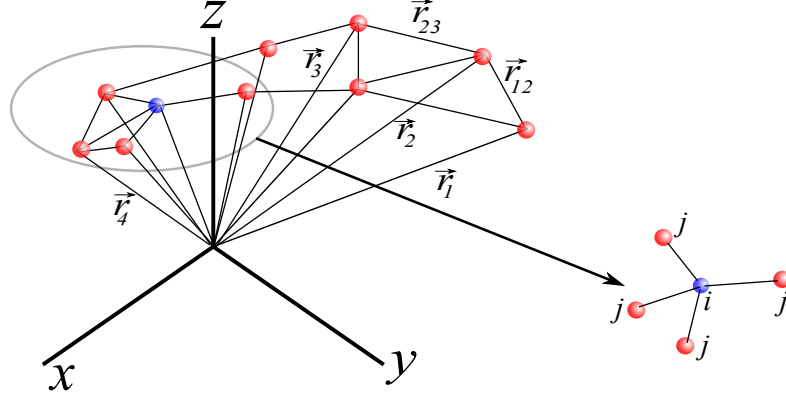


Fig. 4. Geometry of the problem with many galaxies (or other astronomical objects marked by red and blue balls) situated randomly in the Universe. Radius - vectors of those elements (like  $\mathbf{r}_1$ ,  $\mathbf{r}_2$  etc) as well as their separations (like  $\mathbf{r}_{23}$ ) are shown selectively. Blue ball (in the ellipse on the main panel and in the inset) shows the example of  $i$ -th object with the rest being  $j$ -th objects. Division on  $i$  and  $j$  objects is arbitrary and made to calculate the gravitational field, exerted on  $i$ -th object from the rest of the ensemble. In other words, any galaxy can be either of  $i$  or  $j$  type. Inset shows this situation (from the ellipse on the main panel): the gravitational field on the (arbitrary chosen) blue ball  $i$  is a sum of the fields from its neighboring objects  $j$ . The dimensions of the ellipse on the main panel visualize the range of interaction (A28); this range is very long (decays as  $r^4$  so that much more galaxies will be in the range of interaction, but the distant  $j$ -th galaxies make almost zero contribution to the gravity field on  $i$ -th one), it does not have clear boundary but the ellipse gives some guide for eyes. As the number of galaxies is actually infinite and their separations become progressively smaller, the galaxies connecting polyline (i.e. line consisting of all  $\mathbf{r}_{ij}$ ) tends to continuous curve (not shown). In this case all sums are converted to integrals, as described in the text.

The interaction energy  $W_i$  is the energy exerted by the rest of the galaxy ensemble (due to intergalaxy interaction) to the galaxy in the point  $i$ . We can see that after summation (actually integration, see below) over  $\mathbf{r}_j$  the relative intergalaxy distance  $\mathbf{r}_{ij}$  has actually disappeared.

The gradient of the energy (A27) is indeed the gravity field, which acts on  $i$ -th galaxy (or other astronomical object) from the rest  $j$  of these objects ensemble

$$\mathbf{E}_{quad}(\mathbf{r}_i) \equiv \mathbf{E}_{quad,i} = \sum_j m_j \nabla V(\mathbf{r}_j - \mathbf{r}_i) = \mathbf{i}_r E_0 \sum_j m_j \frac{3 \cos^2 \theta_{ij} - 1}{r_{ij}^4}, \quad (\text{A28})$$

which is the expression (2) from the text, rewritten explicitly in terms of vectors  $\mathbf{r}_i$  and  $\mathbf{r}_j$ .

Having the expression (A28), we can write explicitly the distribution function of random quadrupolar fields, Eq. (3) from the text

$$f(\mathbf{E}) = \overline{\delta(\mathbf{E} - \mathbf{E}_i)} \equiv \overline{\delta(\mathbf{E} - \mathbf{E}_{quad}(\mathbf{r}_i))} = \overline{\delta\left(\mathbf{E} - \mathbf{i}_r E_0 \sum_j m_j \frac{3 \cos^2 \theta_{ij} - 1}{r_{ij}^4}\right)}, \quad (\text{A29})$$

where bar means the averaging over random spatial configurations of galaxies and other astronomical objects.

In performing the actual averagings in the expression (A29) (see Fig.4), with respect to the fact that number of galaxies is infinite and their "elementary separations"  $\mathbf{r}_{ij}$  become very small, we can change summations in (A28) and (A29) to integrations using the expression for gravity field  $\mathbf{E}_i$  in the form (2). Further averagings in (A29) are prescribed in the text, see also Ref. (Stephanovich & Godłowski 2015).

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